CHAPTER THREE

HOW TO CALCULATE PRESENT VALUES

IN CHAPTER 2 we learned how to work out the value of an asset that produces cash exactly one year from now. But we did not explain how to value assets that produce cash two years from now or in several future years. That is the first task for this chapter. We will then have a look at some shortcut methods for calculating present values and at some specialized present value formulas. In particular we will show how to value an investment that makes a steady stream of payments forever (a *perpetuity*) and one that produces a steady stream for a limited period (an *annuity*). We will also look at investments that produce a steadily growing stream of payments.

The term *interest rate* sounds straightforward enough, but we will see that it can be defined in various ways. We will first explain the distinction between *compound interest* and *simple interest*. Then we will discuss the difference between the nominal interest rate and the real interest rate. This difference arises because the purchasing power of interest income is reduced by inflation.

By then you will deserve some payoff for the mental investment you have made in learning about present values. Therefore, we will try out the concept on bonds. In Chapter 4 we will look at the valuation of common stocks, and after that we will tackle the firm's capital investment decisions at a practical level of detail.

3.1 VALUING LONG-LIVED ASSETS

Do you remember how to calculate the present value (PV) of an asset that produces a cash flow (C_1) one year from now?

$$\text{PV} = \text{DF}_1 \times C_1 = \frac{C_1}{1 + r_1}$$

The discount factor for the year-1 cash flow is DF_1, and r_1 is the opportunity cost of investing your money for one year. Suppose you will receive a certain cash inflow of \$100 next year ($C_1 = 100$) and the rate of interest on one-year U.S. Treasury notes is 7 percent ($r_1 = .07$). Then present value equals

$$\text{PV} = \frac{C_1}{1 + r_1} = \frac{100}{1.07} = \$93.46$$

The present value of a cash flow two years hence can be written in a similar way as

$$\text{PV} = \text{DF}_2 \times C_2 = \frac{C_2}{(1 + r_2)^2}$$

C_2 is the year-2 cash flow, DF_2 is the discount factor for the year-2 cash flow, and r_2 is the annual rate of interest on money invested for two years. Suppose you get another cash flow of \$100 in year 2 ($C_2 = 100$). The rate of interest on two-year Treasury notes is 7.7 percent per year ($r_2 = .077$); this means that a dollar invested in two-year notes will grow to $1.077^2 = \$1.16$ by the end of two years. The present value of your year-2 cash flow equals

$$\text{PV} = \frac{C_2}{(1 + r_2)^2} = \frac{100}{(1.077)^2} = \$86.21$$

Valuing Cash Flows in Several Periods

One of the nice things about present values is that they are all expressed in current dollars—so that you can add them up. In other words, the present value of cash flow $A + B$ is equal to the present value of cash flow A plus the present value of cash flow B. This happy result has important implications for investments that produce cash flows in several periods.

We calculated above the value of an asset that produces a cash flow of C_1 in year 1, and we calculated the value of another asset that produces a cash flow of C_2 in year 2. Following our additivity rule, we can write down the value of an asset that produces cash flows in *each* year. It is simply

$$PV = \frac{C_1}{1 + r_1} + \frac{C_2}{(1 + r_2)^2}$$

We can obviously continue in this way to find the present value of an extended stream of cash flows:

$$PV = \frac{C_1}{1 + r_1} + \frac{C_2}{(1 + r_2)^2} + \frac{C_3}{(1 + r_3)^3} + \cdots$$

This is called the **discounted cash flow** (or **DCF**) formula. A shorthand way to write it is

$$PV = \sum \frac{C_t}{(1 + r_t)^t}$$

where Σ refers to the sum of the series. To find the *net* present value (NPV) we add the (usually negative) initial cash flow, just as in Chapter 2:

$$NPV = C_0 + PV = C_0 + \sum \frac{C_t}{(1 + r_t)^t}$$

Why the Discount Factor Declines as Futurity Increases—And a Digression on Money Machines

If a dollar tomorrow is worth less than a dollar today, one might suspect that a dollar the day after tomorrow should be worth even less. In other words, the discount factor DF_2 should be less than the discount factor DF_1. But is this *necessarily* so, when there is a different interest rate r_t for each period?

Suppose r_1 is 20 percent and r_2 is 7 percent. Then

$$DF_1 = \frac{1}{1.20} = .83$$

$$DF_2 = \frac{1}{(1.07)^2} = .87$$

Apparently the dollar received the day after tomorrow is *not* necessarily worth less than the dollar received tomorrow.

But there is something wrong with this example. Anyone who could borrow and lend at these interest rates could become a millionaire overnight. Let us see how such a "money machine" would work. Suppose the first person to spot the opportunity is Hermione Kraft. Ms. Kraft first lends $1,000 for one year at 20 percent. That is an attractive enough return, but she notices that there is a way to earn an *immediate* profit on her investment and be ready to play the game again. She reasons as follows. Next year she will have $1,200 which can be reinvested for a further year. Although she does not know what interest rates will be at that time, she does know that she can always put the money in a checking account and be sure of having $1,200 at the end of year 2. Her next step, therefore, is to go to her bank and borrow the present value of this $1,200. At 7 percent interest this present value is

$$PV = \frac{1200}{(1.07)^2} = \$1{,}048$$

Thus Ms. Kraft invests $1,000, borrows back $1,048, and walks away with a profit of $48. If that does not sound like very much, remember that the game can be played again immediately, this time with $1,048. In fact it would take Ms. Kraft only 147 plays to become a millionaire (before taxes).[1]

Of course this story is completely fanciful. Such an opportunity would not last long in capital markets like ours. Any bank that would allow you to lend for one year at 20 percent and borrow for two years at 7 percent would soon be wiped out by a rush of small investors hoping to become millionaires and a rush of millionaires hoping to become billionaires. There are, however, two lessons to our story. The first is that a dollar tomorrow *cannot* be worth less than a dollar the day after tomorrow. In other words, the value of a dollar received at the end of one year (DF_1) must be greater than the value of a dollar received at the end of two years (DF_2). There must be some extra gain[2] from lending for two periods rather than one: $(1 + r_2)^2$ must be greater than $1 + r_1$.

Our second lesson is a more general one and can be summed up by the precept "There is no such thing as a money machine."[3] In well-functioning capital markets, any potential money machine will be eliminated almost instantaneously by investors who try to take advantage of it. Therefore, beware of self-styled experts who offer you a chance to participate in a sure thing.

Later in the book we will invoke the *absence* of money machines to prove several useful properties about security prices. That is, we will make statements like "The prices of securities X and Y must be in the following relationship—otherwise there would be a money machine and capital markets would not be in equilibrium."

Ruling out money machines does not require that interest rates be the same for each future period. This relationship between the interest rate and the maturity of the cash flow is called the **term structure of interest rates.** We are going to look at term structure in Chapter 24, but for now we will finesse the issue by assuming that the term structure is "flat"—in other words, the interest rate is the same regardless of the date of the cash flow. This means that we can replace the series of interest rates $r_1, r_2, \ldots, r_t$, etc., with a single rate r and that we can write the present value formula as

$$PV = \frac{C_1}{1 + r} + \frac{C_2}{(1 + r)^2} + \cdots$$

[1]That is, $1{,}000 \times (1.04813)^{147} = \$1{,}002{,}000$.

[2]The extra return for lending two years rather than one is often referred to as a *forward rate of return*. Our rule says that the forward rate cannot be negative.

[3]The technical term for money machine is *arbitrage*. There are no opportunities for arbitrage in well-functioning capital markets.

Calculating PVs and NPVs

You have some bad news about your office building venture (the one described at the start of Chapter 2). The contractor says that construction will take two years instead of one and requests payment on the following schedule:

1. A $100,000 down payment now. (Note that the land, worth $50,000, must also be committed now.)
2. A $100,000 progress payment after one year.
3. A final payment of $100,000 when the building is ready for occupancy at the end of the second year.

Your real estate adviser maintains that despite the delay the building will be worth $400,000 when completed.

All this yields a new set of cash-flow forecasts:

Period	t = 0	t = 1	t = 2
Land	−50,000		
Construction	−100,000	−100,000	−100,000
Payoff			+400,000
Total	$C_0 = -150{,}000$	$C_1 = -100{,}000$	$C_2 = +300{,}000$

If the interest rate is 7 percent, then NPV is

$$\begin{aligned} \text{NPV} &= C_0 + \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} \\ &= -150{,}000 - \frac{100{,}000}{1.07} + \frac{300{,}000}{(1.07)^2} \end{aligned}$$

Table 3.1 calculates NPV step by step. The calculations require just a few keystrokes on an electronic calculator. Real problems can be much more complicated, however, so financial managers usually turn to calculators especially programmed for present value calculations or to spreadsheet programs on personal computers. In some cases it can be convenient to look up discount factors in present value tables like Appendix Table 1 at the end of this book.

Fortunately the news about your office venture is not all bad. The contractor is willing to accept a delayed payment; this means that the present value of the contractor's fee is less than before. This partly offsets the delay in the payoff. As Table 3.1 shows,

TABLE 3.1

Present value worksheet.

Period	Discount Factor	Cash Flow	Present Value
0	1.0	−150,000	−150,000
1	$\frac{1}{1.07} = .935$	−100,000	−93,500
2	$\frac{1}{(1.07)^2} = .873$	+300,000	+261,900
			Total = NPV = $18,400

the net present value is \$18,400—not a substantial decrease from the \$23,800 calculated in Chapter 2. Since the net present value is positive, you should still go ahead.[4]

3.2 LOOKING FOR SHORTCUTS—PERPETUITIES AND ANNUITIES

Sometimes there are shortcuts that make it easy to calculate present values. Let us look at some examples.

Among the securities that have been issued by the British government are so-called **perpetuities.** These are bonds that the government is under no obligation to repay but that offer a fixed income for each year to perpetuity. The annual rate of return on a perpetuity is equal to the promised annual payment divided by the present value:

$$\text{Return} = \frac{\text{cash flow}}{\text{present value}}$$

$$r = \frac{C}{\text{PV}}$$

We can obviously twist this around and find the present value of a perpetuity given the discount rate r and the cash payment C. For example, suppose that some worthy person wishes to endow a chair in finance at a business school with the initial payment occurring at the end of the first year. If the rate of interest is 10 percent and if the aim is to provide \$100,000 a year in perpetuity, the amount that must be set aside today is[5]

$$\text{Present value of perpetuity} = \frac{C}{r} = \frac{100{,}000}{.10} = \$1{,}000{,}000$$

How to Value Growing Perpetuities

Suppose now that our benefactor suddenly recollects that no allowance has been made for growth in salaries, which will probably average about 4 percent a year starting in year 1. Therefore, instead of providing \$100,000 a year in perpetuity, the benefactor must provide \$100,000 in year 1, $1.04 \times \$100{,}000$ in year 2, and so on. If

[4]We assume the cash flows are safe. If they are risky forecasts, the opportunity cost of capital could be higher, say 12 percent. NPV at 12 percent is just about zero.

[5]You can check this by writing down the present value formula

$$\text{PV} = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots$$

Now let $C/(1 + r) = a$ and $1/(1 + r) = x$. Then we have (1) $\text{PV} = a(1 + x + x^2 + \cdots)$.
Multiplying both sides by x, we have (2) $\text{PV}x = a(x + x^2 + \cdots)$.
Subtracting (2) from (1) gives us $\text{PV}(1 - x) = a$. Therefore, substituting for a and x,

$$\text{PV}\left(1 - \frac{1}{1+r}\right) = \frac{C}{1+r}$$

Multiplying both sides by $(1 + r)$ and rearranging gives

$$\text{PV} = \frac{C}{r}$$

we call the growth rate in salaries g, we can write down the present value of this stream of cash flows as follows:

$$\begin{aligned} PV &= \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \frac{C_3}{(1+r)^3} + \cdots \\ &= \frac{C_1}{1+r} + \frac{C_1(1+g)}{(1+r)^2} + \frac{C_1(1+g)^2}{(1+r)^3} + \cdots \end{aligned}$$

Fortunately, there is a simple formula for the sum of this geometric series.[6] If we assume that r is greater than g, our clumsy-looking calculation simplifies to

$$\text{Present value of growing perpetuity} = \frac{C_1}{r-g}$$

Therefore, if our benefactor wants to provide perpetually an annual sum that keeps pace with the growth rate in salaries, the amount that must be set aside today is

$$PV = \frac{C_1}{r-g} = \frac{100{,}000}{.10 - .04} = \$1{,}666{,}667$$

How to Value Annuities

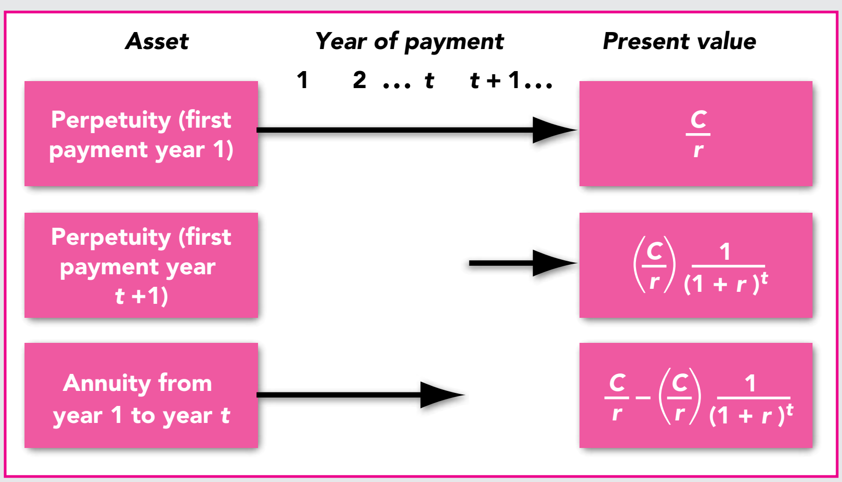
An **annuity** is an asset that pays a fixed sum each year for a specified number of years. The equal-payment house mortgage or installment credit agreement are common examples of annuities.

Figure 3.1 illustrates a simple trick for valuing annuities. The first row represents a perpetuity that produces a cash flow of C in each year beginning in year 1. It has a present value of

$$PV = \frac{C}{r}$$

FIGURE 3.1

An annuity that makes payments in each of years 1 to t is equal to the difference between two perpetuities.



[6] We need to calculate the sum of an infinite geometric series $PV = a(1 + x + x^2 + \cdots)$ where $a = C_1/(1 + r)$ and $x = (1 + g)/(1 + r)$. In footnote 5 we showed that the sum of such a series is $a/(1 - x)$. Substituting for a and x in this formula,

$$PV = \frac{C_1}{r-g}$$

The second row represents a second *perpetuity* that produces a cash flow of C in each year *beginning in year* $t + 1$. It *will* have a present value of C/r in year t and it therefore has a present value today of

$$\text{PV} = \frac{C}{r(1 + r)^t}$$

Both perpetuities provide a cash flow from year $t + 1$ onward. The only difference between the two perpetuities is that the first one *also* provides a cash flow in each of the years 1 through t. In other words, the difference between the two perpetuities is an annuity of C for t years. The present value of this annuity is, therefore, the difference between the values of the two perpetuities:

$$\text{Present value of annuity} = C\left[\frac{1}{r} - \frac{1}{r(1 + r)^t}\right]$$

The expression in brackets is the *annuity factor,* which is the present value at discount rate r of an annuity of \$1 paid at the end of each of t periods.[7]

Suppose, for example, that our benefactor begins to vacillate and wonders what it would cost to endow a chair providing \$100,000 a year for only 20 years. The answer calculated from our formula is

$$\text{PV} = 100{,}000\left[\frac{1}{.10} - \frac{1}{.10(1.10)^{20}}\right] = 100{,}000 \times 8.514 = \$851{,}400$$

Alternatively, we can simply look up the answer in the annuity table in the Appendix at the end of the book (Appendix Table 3). This table gives the present value of a dollar to be received in each of t periods. In our example $t = 20$ and the interest rate $r = .10$, and therefore we look at the twentieth number from the top in the *10 percent* column. It is 8.514. Multiply 8.514 by \$100,000, and we have our answer, \$851,400.

Remember that the annuity formula assumes that the first payment occurs one period hence. If the first cash payment occurs immediately, we would need to discount each cash flow by one less year. So the present value would be increased by the multiple $(1 + r)$. For example, if our benefactor were prepared to make 20 annual payments *starting immediately,* the value would be \$851,400 $\times$ 1.10 = \$936,540. An annuity offering an immediate payment is known as an *annuity due.*

[7]Again we can work this out from first principles. We need to calculate the sum of the finite geometric series (1) $\text{PV} = a(1 + x + x^2 + \cdots + x^{t-1})$,

where $a = C/(1 + r)$ and $x = 1/(1 + r)$.

Multiplying both sides by x, we have (2) $\text{PV}x = a(x + x^2 + \cdots + x^t)$.

Subtracting (2) from (1) gives us $\text{PV}(1 - x) = a(1 - x^t)$.

Therefore, substituting for a and x,

$$\text{PV}\left(1 - \frac{1}{1 + r}\right) = C\left[\frac{1}{1 + r} - \frac{1}{(1 + r)^{t+1}}\right]$$

Multiplying both sides by $(1 + r)$ and rearranging gives

$$\text{PV} = C\left[\frac{1}{r} - \frac{1}{r(1 + r)^t}\right]$$

You should always be on the lookout for ways in which you can use these formulas to make life easier. For example, we sometimes need to calculate how much a series of annual payments earning a fixed annual interest rate would amass to by the end of t periods. In this case it is easiest to calculate the *present* value, and then multiply it by $(1 + r)^t$ to find the future value.[8] Thus suppose our benefactor wished to know how much wealth $100,000 would produce if it were invested each year instead of being given to those no-good academics. The answer would be

$$\text{Future value} = \text{PV} \times 1.10^{20} = \$851{,}400 \times 6.727 = \$5.73 \text{ million}$$

How did we know that 1.10^{20} was 6.727? Easy—we just looked it up in Appendix Table 2 at the end of the book: "Future Value of $1 at the End of t Periods."

3.3 COMPOUND INTEREST AND PRESENT VALUES

There is an important distinction between **compound interest** and **simple interest.** When money is invested at compound interest, each interest payment is reinvested to earn more interest in subsequent periods. In contrast, the opportunity to earn interest on interest is not provided by an investment that pays only simple interest.

Table 3.2 compares the growth of $100 invested at compound versus simple interest. Notice that in the simple interest case, *the interest is paid only on the initial in-*

	Simple Interest					Compound Interest				
Year	Starting Balance	+	Interest	=	Ending Balance	Starting Balance	+	Ending Interest	=	Balance
1	100	+	10	=	110	100	+	10	=	110
2	110	+	10	=	120	110	+	11	=	121
3	120	+	10	=	130	121	+	12.1	=	133.1
4	130	+	10	=	140	133.1	+	13.3	=	146.4
10	190	+	10	=	200	236	+	24	=	259
20	290	+	10	=	300	612	+	61	=	673
50	590	+	10	=	600	10,672	+	1,067	=	11,739
100	1,090	+	10	=	1,100	1,252,783	+	125,278	=	1,378,061
200	2,090	+	10	=	2,100	17,264,116,042	+	1,726,411,604	=	18,990,527,646
226	2,350	+	10	=	2,360	205,756,782,755	+	20,575,678,275	=	226,332,461,030

TABLE 3.2

Value of $100 invested at 10 percent simple and compound interest.

[8] For example, suppose you receive a cash flow of C in year 6. If you invest this cash flow at an interest rate of r, you will have by year 10 an investment worth $C(1 + r)^4$. You can get the same answer by calculating the *present value* of the cash flow $\text{PV} = C/(1 + r)^6$ and then working out how much you would have by year 10 if you invested this sum today:

$$\text{Future value} = \text{PV}(1 + r)^{10} = \frac{C}{(1 + r)^6} \times (1 + r)^{10} = C(1 + r)^4$$

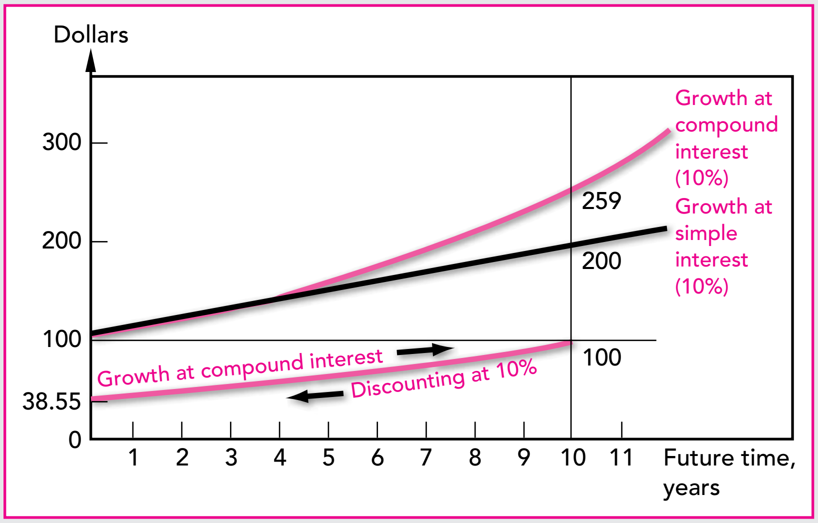


FIGURE 3.2

Compound interest versus simple interest. The top two ascending lines show the growth of $100 invested at simple and compound interest. The longer the funds are invested, the greater the advantage with compound interest. The bottom line shows that $38.55 must be invested now to obtain $100 after 10 periods. Conversely, the present value of $100 to be received after 10 years is $38.55.

vestment of $100. Your wealth therefore increases by just $10 a year. In the compound interest case, you earn 10 percent on your initial investment in the first year, which gives you a balance at the end of the year of $100 \times 1.10 = \$110$. Then in the second year you earn 10 percent on this $110, which gives you a balance at the end of the second year of $100 \times 1.10^2 = \$121$.

Table 3.2 shows that the difference between simple and compound interest is nil for a one-period investment, trivial for a two-period investment, but overwhelming for an investment of 20 years or more. A sum of $100 invested during the American Revolution and earning compound interest of 10 percent a year would now be worth over $226 billion. If only your ancestors could have put away a few cents.

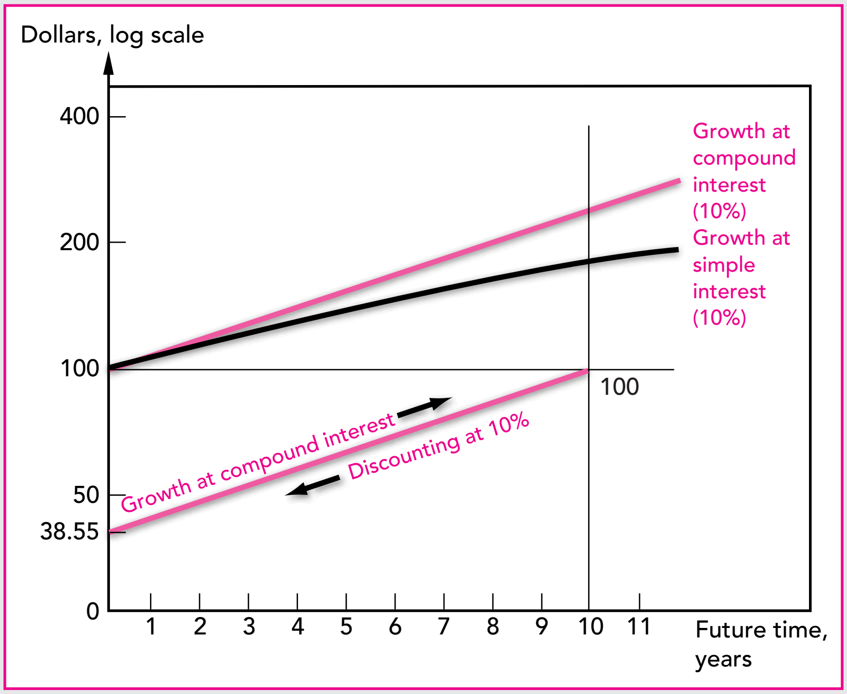
The two top lines in Figure 3.2 compare the results of investing $100 at 10 percent simple interest and at 10 percent compound interest. It looks as if the rate of growth is constant under simple interest and accelerates under compound interest. However, this is an optical illusion. We know that under compound interest our wealth grows at a *constant* rate of 10 percent. Figure 3.3 is in fact a more useful presentation. Here the numbers are plotted on a semilogarithmic scale and the constant compound growth rates show up as straight lines.

Problems in finance almost always involve compound interest rather than simple interest, and therefore financial people always assume that you are talking about compound interest unless you specify otherwise. Discounting is a process of compound interest. Some people find it intuitively helpful to replace the question, What is the present value of $100 to be received 10 years from now, if the opportunity cost of capital is 10 percent? with the question, How much would I have to invest now in order to receive $100 after 10 years, given an interest rate of 10 percent? The answer to the first question is

$$PV = \frac{100}{(1.10)^{10}} = \$38.55$$

FIGURE 3.3

The same story as Figure 3.2, except that the vertical scale is logarithmic. A constant compound rate of growth means a straight ascending line. This graph makes clear that the growth rate of funds invested at simple interest actually *declines* as time passes.



And the answer to the second question is

$$\text{Investment} \times (1.10)^{10} = \$100$$

$$\text{Investment} = \frac{100}{(1.10)^{10}} = \$38.55$$

The bottom lines in Figures 3.2 and 3.3 show the growth path of an initial investment of \$38.55 to its terminal value of \$100. One can think of discounting as traveling *back* along the bottom line, from future value to present value.

A Note on Compounding Intervals

So far we have implicitly assumed that each cash flow occurs at the end of the year. This is sometimes the case. For example, in France and Germany most corporations pay interest on their bonds annually. However, in the United States and Britain most pay interest semiannually. In these countries, the investor can earn an additional six months' interest on the first payment, so that an investment of \$100 in a bond that paid interest of 10 percent per annum compounded semiannually would amount to \$105 after the first six months, and by the end of the year it would amount to $1.05^2 \times 100 = \$110.25$. In other words, 10 percent compounded semiannually is equivalent to 10.25 percent compounded annually.

Let's take another example. Suppose a bank makes automobile loans requiring monthly payments at an *annual percentage rate (APR)* of 6 percent per year. What does that mean, and what is the true rate of interest on the loans?

With monthly payments, the bank charges one-twelfth of the APR in each month, that is, $6/12 = .5$ percent. Because the monthly return is compounded, the bank actually earns more than 6 percent per year. Suppose that the bank starts with \$10 million of automobile loans outstanding. This investment grows to $\$10 \times 1.005 = \$10.05$ million after month 1, to $\$10 \times 1.005^2 = \$10.10025$ million after month 2, and to $\$10 \times 1.005^{12} = \$10.61678$ million after 12 months.[9] Thus the bank is quoting a 6 percent APR but actually earns 6.1678 percent if interest payments are made monthly.[10]

In general, an investment of \$1 at a rate of r per annum compounded m times a year amounts by the end of the year to $[1 + (r/m)]^m$, and the equivalent annually compounded rate of interest is $[1 + (r/m)]^m - 1$.

Continuous Compounding The attractions to the investor of more frequent payments did not escape the attention of the savings and loan companies in the 1960s and 1970s. Their rate of interest on deposits was traditionally stated as an annually compounded rate. The government used to stipulate a maximum annual rate of interest that could be paid but made no mention of the compounding interval. When interest ceilings began to pinch, savings and loan companies changed progressively to semiannual and then to monthly compounding. Therefore the equivalent annually compounded rate of interest increased first to $[1 + (r/2)]^2 - 1$ and then to $[1 + (r/12)]^{12} - 1$.

Eventually one company quoted a **continuously compounded rate,** so that payments were assumed to be spread evenly and continuously throughout the year. In terms of our formula, this is equivalent to letting m approach infinity.[11] This might seem like a lot of calculations for the savings and loan companies. Fortunately, however, someone remembered high school algebra and pointed out that as m approaches infinity $[1 + (r/m)]^m$ approaches $(2.718)^r$. The figure 2.718—or e, as it is called—is simply the base for natural logarithms.

One dollar invested at a continuously compounded rate of r will, therefore, grow to $e^r = (2.718)^r$ by the end of the first year. By the end of t years it will grow to $e^{rt} = (2.718)^{rt}$. Appendix Table 4 at the end of the book is a table of values of e^{rt}. Let us practice using it.

Example 1 Suppose you invest \$1 at a continuously compounded rate of 11 percent ($r = .11$) for one year ($t = 1$). The end-year value is $e^{.11}$, which you can see from the second row of Appendix Table 4 is \$1.116. In other words, investing at 11 percent a year *continuously* compounded is exactly the same as investing at 11.6 percent a year *annually* compounded.

Example 2 Suppose you invest \$1 at a continuously compounded rate of 11 percent ($r = .11$) for two years ($t = 2$). The final value of the investment is $e^{rt} = e^{.22}$. You can see from the third row of Appendix Table 4 that $e^{.22}$ is \$1.246.

[9]Individual borrowers gradually pay off their loans. We are assuming that the aggregate amount loaned by the bank to all its customers stays constant at \$10 million.

[10]Unfortunately, U.S. truth-in-lending laws require lenders to quote interest rates for most types of consumer loans as APRs rather than true annual rates.

[11]When we talk about *continuous* payments, we are pretending that money can be dispensed in a continuous stream like water out of a faucet. One can never quite do this. For example, instead of paying out \$100,000 every year, our benefactor could pay out \$100 every 8¾ hours or \$1 every 5¼ minutes or 1 cent every 3⅛ seconds but could not pay it out *continuously.* Financial managers *pretend* that payments are continuous rather than hourly, daily, or weekly because (1) it simplifies the calculations, and (2) it gives a *very* close approximation to the NPV of frequent payments.

There is a particular value to continuous compounding in capital budgeting, where it may often be more reasonable to assume that a cash flow is spread evenly over the year than that it occurs at the year's end. It is easy to adapt our previous formulas to handle this. For example, suppose that we wish to compute the present value of a perpetuity of C dollars a year. We already know that if the payment is made at the end of the year, we divide the payment by the *annually* compounded rate of r:

$$\text{PV} = \frac{C}{r}$$

If the same total payment is made in an even stream throughout the year, we use the same formula but substitute the *continuously* compounded rate.

Example 3 Suppose the annually compounded rate is 18.5 percent. The present value of a \$100 perpetuity, with each cash flow received at the end of the year, is 100/.185 = \$540.54. If the cash flow is received continuously, we must divide \$100 by 17 percent, because 17 percent continuously compounded is equivalent to 18.5 percent annually compounded ($e^{.17} = 1.185$). The present value of the continuous cash flow stream is 100/.17 = \$588.24.

For any other continuous payments, we can always use our formula for valuing annuities. For instance, suppose that our philanthropist has thought more seriously and decided to found a home for elderly donkeys, which will cost \$100,000 a year, starting immediately, and spread evenly over 20 years. Previously, we used the annually compounded rate of 10 percent; now we must use the continuously compounded rate of $r = 9.53$ percent ($e^{.0953} = 1.10$). To cover such an expenditure, then, our philanthropist needs to set aside the following sum:[12]

$$\begin{aligned}\text{PV} &= C\left(\frac{1}{r} - \frac{1}{r} \times \frac{1}{e^{rt}}\right) \\ &= 100{,}000\left(\frac{1}{.0953} - \frac{1}{.0953} \times \frac{1}{6.727}\right) = 100{,}000 \times 8.932 = \$893{,}200\end{aligned}$$

Alternatively, we could have cut these calculations short by using Appendix Table 5. This shows that, if the annually compounded return is 10 percent, then \$1 a year spread over 20 years is worth \$8.932.

If you look back at our earlier discussion of annuities, you will notice that the present value of \$100,000 paid at the *end* of each of the 20 years was \$851,400.

[12]Remember that an annuity is simply the difference between a perpetuity received today and a perpetuity received in year t. A continuous stream of C dollars a year in perpetuity is worth C/r, where r is the continuously compounded rate. Our annuity, then, is worth

$$\text{PV} = \frac{C}{r} - \text{present value of } \frac{C}{r} \text{ received in year } t$$

Since r is the continuously compounded rate, C/r received in year t is worth $(C/r) \times (1/e^{rt})$ today. Our annuity formula is therefore

$$\text{PV} = \frac{C}{r} - \frac{C}{r} \times \frac{1}{e^{rt}}$$

sometimes written as

$$\frac{C}{r}(1 - e^{-rt})$$

Therefore, it costs the philanthropist $41,800—or 5 percent—more to provide a continuous payment stream.

Often in finance we need only a ballpark estimate of present value. An error of 5 percent in a present value calculation may be perfectly acceptable. In such cases it doesn't usually matter whether we assume that cash flows occur at the end of the year or in a continuous stream. At other times precision matters, and we do need to worry about the exact frequency of the cash flows.

3.4 NOMINAL AND REAL RATES OF INTEREST

If you invest $1,000 in a bank deposit offering an interest rate of 10 percent, the bank promises to pay you $1,100 at the end of the year. But it makes no promises about what the $1,100 will buy. That will depend on the rate of inflation over the year. If the prices of goods and services increase by more than 10 percent, you have lost ground in terms of the goods that you can buy.

Several indexes are used to track the general level of prices. The best known is the Consumer Price Index, or CPI, which measures the number of dollars that it takes to pay for a typical family's purchases. The change in the CPI from one year to the next measures the rate of inflation. Figure 3.4 shows the rate of inflation in the United

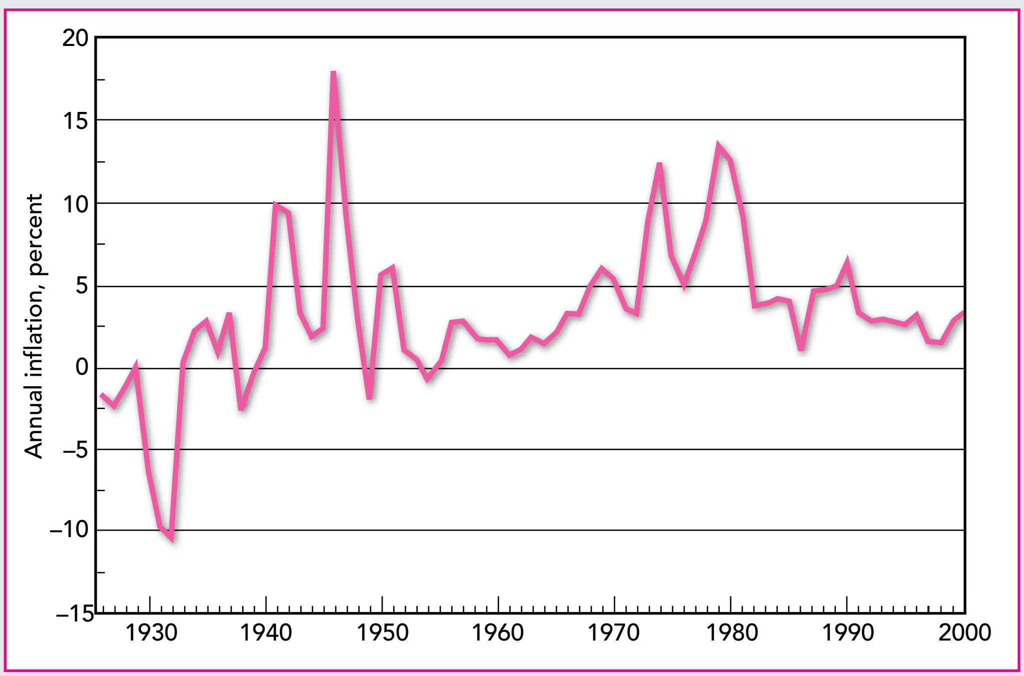


FIGURE 3.4

Annual rates of inflation in the United States from 1926 to 2000.

Source: Ibbotson Associates, Inc., *Stocks, Bonds, Bills, and Inflation,* 2001 Yearbook, Chicago, 2001.

States since 1926. During the Great Depression there was actual *de*flation; prices of goods on average fell. Inflation touched a peak just after World War II, when it reached 18 percent. This figure, however, pales into insignificance compared with inflation in Yugoslavia in 1993, which at its peak was almost 60 percent *a day.*

Economists sometimes talk about current, or nominal, dollars versus constant, or real, dollars. For example, the *nominal* cash flow from your one-year bank deposit is $1,100. But suppose prices of goods rise over the year by 6 percent; then each dollar will buy you 6 percent less goods next year than it does today. So at the end of the year $1,100 will buy the same quantity of goods as 1,100/1.06 = $1,037.74 today. The nominal payoff on the deposit is $1,100, but the *real* payoff is only $1,037.74.

The general formula for converting nominal cash flows at a future period t to real cash flows is

$$\text{Real cash flow} = \frac{\text{nominal cash flow}}{(1 + \text{inflation rate})^t}$$

For example, if you were to invest that $1,000 for 20 years at 10 percent, your future nominal payoff would be $1{,}000 \times 1.1^{20} = \$6{,}727.50$, but with an inflation rate of 6 percent a year, the real value of that payoff would be $6{,}727.50/1.06^{20} = \$2{,}097.67$. In other words, you will have roughly six times as many dollars as you have today, but you will be able to buy only twice as many goods.

When the bank quotes you a 10 percent interest rate, it is quoting a nominal interest rate. The rate tells you how rapidly your money will grow:

Invest Current Dollars		Receive Period-1 Dollars	Result
1,000	→	1,100	10% *nominal* rate of return

However, with an inflation rate of 6 percent you are only 3.774 percent better off at the end of the year than at the start:

Invest Current Dollars		Expected Real Value of Period-1 Receipts	Result
1,000	→	1,037.74	3.774% expected *real* rate of return

Thus, we could say, "The bank account offers a 10 percent nominal rate of return," or "It offers a 3.774 percent expected real rate of return." Note that the nominal rate is certain but the real rate is only expected. The actual real rate cannot be calculated until the end of the year arrives and the inflation rate is known.

The 10 percent nominal rate of return, with 6 percent inflation, translates into a 3.774 percent real rate of return. The formula for calculating the real rate of return is

$$\begin{aligned} 1 + r_{\text{nominal}} &= (1 + r_{\text{real}})(1 + \text{inflation rate}) \\ &= 1 + r_{\text{real}} + \text{inflation rate} + (r_{\text{real}})(\text{inflation rate}) \end{aligned}$$

In our example,

$$1.10 = 1.03774 \times 1.06$$

3.5 USING PRESENT VALUE FORMULAS TO VALUE BONDS

When governments or companies borrow money, they often do so by issuing bonds. A bond is simply a long-term debt. If you own a bond, you receive a fixed set of cash payoffs: Each year until the bond matures, you collect an interest payment; then at maturity, you also get back the face value of the bond. The face value of the bond is known as the *principal.* Therefore, when the bond matures, the government pays you principal and interest.

If you want to buy or sell a bond, you simply contact a bond dealer, who will quote a price at which he or she is prepared to buy or sell. Suppose, for example, that in June 2001 you invested in a 7 percent 2006 U.S. Treasury bond. The bond has a coupon rate of 7 percent and a face value of $1,000. This means that each year until 2006 you will receive an interest payment of $.07 \times 1{,}000 = \$70$. The bond matures in May 2006. At that time the Treasury pays you the final $70 interest, plus the $1,000 face value. So the cash flows from owning the bond are as follows:

Cash Flows ($)				
2002	**2003**	**2004**	**2005**	**2006**
70	70	70	70	1,070

What is the present value of these payoffs? To determine that, we need to look at the return provided by similar securities. Other medium-term U.S. Treasury bonds in the summer of 2001 offered a return of about 4.8 percent. That is what investors were giving up when they bought the 7 percent Treasury bonds. Therefore to value the 7 percent bonds, we need to discount the cash flows at 4.8 percent:

$$\text{PV} = \frac{70}{1.048} + \frac{70}{(1.048)^2} + \frac{70}{(1.048)^3} + \frac{70}{(1.048)^4} + \frac{1070}{(1.048)^5} = 1{,}095.78$$

Bond prices are usually expressed as a percentage of the face value. Thus, we can say that our 7 percent Treasury bond is worth $1,095.78, or 109.578 percent.

You may have noticed a shortcut way to value the Treasury bond. The bond is like a package of two investments: The first investment consists of five annual coupon payments of $70 each, and the second investment is the payment of the $1,000 face value at maturity. Therefore, you can use the annuity formula to value the coupon payments and add on the present value of the final payment:

$$\begin{aligned}\text{PV(bond)} &= \text{PV(coupon payments)} + \text{PV(final payment)}\\ &= (\text{coupon} \times \text{five-year annuity factor}) + \\ &\quad\ (\text{final payment} \times \text{discount factor})\\ &= 70\left[\frac{1}{.048} - \frac{1}{.048(1.048)^5}\right] + \frac{1000}{1.048^5} = 304.75 + 791.03 = 1095.78\end{aligned}$$

Any Treasury bond can be valued as a package of an annuity (the coupon payments) and a single payment (the repayment of the face value).

Rather than asking the value of the bond, we could have phrased our question the other way around: If the price of the bond is $1,095.78, what return do investors expect? In that case, we need to find the value of r that solves the following equation:

$$1095.78 = \frac{70}{1+r} + \frac{70}{(1+r)^2} + \frac{70}{(1+r)^3} + \frac{70}{(1+r)^4} + \frac{1070}{(1+r)^5}$$

The rate r is often called the bond's **yield to maturity.** In our case r is 4.8 percent. If you discount the cash flows at 4.8 percent, you arrive at the bond's price of $1,095.78. The only *general* procedure for calculating the yield to maturity is trial and error, but spreadsheet programs or specially programmed electronic calculators will usually do the trick.

You may have noticed that the formula that we used for calculating the present value of 7 percent Treasury bonds was slightly different from the general present value formula that we developed in Section 3.1, where we allowed r_1, the rate of return offered by the capital market on one-year investments, to differ from r_2, the rate of return offered on two-year investments. Then we finessed this problem by assuming that r_1 was the same as r_2. In valuing our Treasury bond, we again assume that investors use the same rate to discount cash flows occurring in different years. That does not matter as long as the term structure is flat, with short-term rates approximately the same as long-term rates. But when the term structure is not flat, professional bond investors discount each cash flow at a different rate. There will be more about that in Chapter 24.

What Happens When Interest Rates Change?

Interest rates fluctuate. In 1945 United States government bonds were yielding less than 2 percent, but by 1981 yields were a touch under 15 percent. International differences in interest rates can be even more dramatic. As we write this in the summer of 2001, short-term interest rates in Japan are less than .2 percent, while in Turkey they are over 60 percent.[13]

How do changes in interest rates affect bond prices? If bond yields in the United States fell to 2 percent, the price of our 7 percent Treasuries would rise to

$$\text{PV} = \frac{70}{1.02} + \frac{70}{(1.02)^2} + \frac{70}{(1.02)^3} + \frac{70}{(1.02)^4} + \frac{1070}{(1.02)^5} = \$1{,}235.67$$

If yields jumped to 10 percent, the price would fall to

$$\text{PV} = \frac{70}{1.10} + \frac{70}{(1.10)^2} + \frac{70}{(1.10)^3} + \frac{70}{(1.10)^4} + \frac{1070}{(1.10)^5} = \$886.28$$

Not surprisingly, the higher the interest rate that investors demand, the less that they will be prepared to pay for the bond.

Some bonds are more affected than others by a change in the interest rate. The effect is greatest when the cash flows on the bond last for many years. The effect is trivial if the bond matures tomorrow.

Compounding Intervals and Bond Prices

In calculating the value of the 7 percent Treasury bonds, we made two approximations. First, we assumed that interest payments occurred annually. In practice,

[13]Early in 2001 the Turkish overnight rate exceeded 20,000 percent.

most U.S. bonds make coupon payments *semiannually,* so that instead of receiving \$70 every year, an investor holding 7 percent bonds would receive \$35 every *half* year. Second, yields on U.S. bonds are usually quoted as semiannually compounded yields. In other words, if the semiannually compounded yield is quoted as 4.8 percent, the yield over six months is 4.8/2 = 2.4 percent.

Now we can recalculate the value of the 7 percent Treasury bonds, recognizing that there are 10 six-month coupon payments of \$35 and a final payment of the \$1,000 face value:

$$\text{PV} = \frac{35}{1.024} + \frac{35}{(1.024)^2} + \cdots + \frac{35}{(1.024)^9} + \frac{1035}{(1.024)^{10}} = \$1{,}096.77$$

SUMMARY

The difficult thing in any present value exercise is to set up the problem correctly. Once you have done that, you must be able to do the calculations, but they are not difficult. Now that you have worked through this chapter, all you should need is a little practice.

The basic present value formula for an asset that pays off in several periods is the following obvious extension of our one-period formula:

$$\text{PV} = \frac{C_1}{1 + r_1} + \frac{C_2}{(1 + r_2)^2} + \cdots$$

You can always work out any present value using this formula, but when the interest rates are the same for each maturity, there may be some shortcuts that can reduce the tedium. We looked at three such cases. The first is an asset that pays C dollars a year in perpetuity. Its present value is simply

$$\text{PV} = \frac{C}{r}$$

The second is an asset whose payments increase at a steady rate g in perpetuity. Its present value is

$$\text{PV} = \frac{C_1}{r - g}$$

The third is an annuity that pays C dollars a year for t years. To find its present value we take the difference between the values of two perpetuities:

$$\text{PV} = C\left[\frac{1}{r} - \frac{1}{r(1 + r)^t}\right]$$

Our next step was to show that discounting is a process of compound interest. Present value is the amount that we would have to invest now at compound interest r in order to produce the cash flows C_1, C_2, etc. When someone offers to lend us a dollar at an annual rate of r, we should always check how frequently the interest is to be compounded. If the compounding interval is annual, we will have to repay $(1 + r)^t$ dollars; on the other hand, if the compounding period is continuous, we will have to repay 2.718^{rt} (or, as it is usually expressed, e^{rt}) dollars. In capital budgeting we often assume that the cash flows occur at the end of each year, and therefore we discount them at an annually compounded rate of interest.

Sometimes, however, it may be better to assume that they are spread evenly over the year; in this case we must make use of continuous compounding.

It is important to distinguish between *nominal* cash flows (the actual number of dollars that you will pay or receive) and *real* cash flows, which are adjusted for inflation. Similarly, an investment may promise a high *nominal* interest rate, but, if inflation is also high, the *real* interest rate may be low or even negative.

We concluded the chapter by applying discounted cash flow techniques to value United States government bonds with fixed annual coupons.

We introduced in this chapter two very important ideas which we will come across several times again. The first is that you can add present values: If your formula for the present value of $A + B$ is not the same as your formula for the present value of A plus the present value of B, you have made a mistake. The second is the notion that there is no such thing as a money machine: If you think you have found one, go back and check your calculations.

FURTHER READING

The material in this chapter should cover all you need to know about the mathematics of discounting; but if you wish to dig deeper, there are a number of books on the subject. Try, for example:

R. Cissell, H. Cissell, and D. C. Flaspohler: *The Mathematics of Finance*, 8th ed., Houghton Mifflin Company, Boston, 1990.

QUIZ

1. At an interest rate of 12 percent, the six-year discount factor is .507. How many dollars is $.507 worth in six years if invested at 12 percent?

2. If the PV of $139 is $125, what is the discount factor?

3. If the eight-year discount factor is .285, what is the PV of $596 received in eight years?

4. If the cost of capital is 9 percent, what is the PV of $374 paid in year 9?

5. A project produces the following cash flows:

Year	Flow
1	432
2	137
3	797

If the cost of capital is 15 percent, what is the project's PV?

6. If you invest $100 at an interest rate of 15 percent, how much will you have at the end of eight years?

7. An investment costs $1,548 and pays $138 in perpetuity. If the interest rate is 9 percent, what is the NPV?

8. A common stock will pay a cash dividend of $4 next year. After that, the dividends are expected to increase indefinitely at 4 percent per year. If the discount rate is 14 percent, what is the PV of the stream of dividend payments?

9. You win a lottery with a prize of $1.5 million. Unfortunately the prize is paid in 10 annual installments. The first payment is next year. How much is the prize really worth? The discount rate is 8 percent.

10. Do not use the Appendix tables for these questions. The interest rate is 10 percent.
    a. What is the PV of an asset that pays $1 a year in perpetuity?
    b. The value of an asset that appreciates at 10 percent per annum approximately doubles in seven years. What is the approximate PV of an asset that pays $1 a year in perpetuity beginning in year 8?
    c. What is the approximate PV of an asset that pays $1 a year for each of the next seven years?
    d. A piece of land produces an income that grows by 5 percent per annum. If the first year's flow is $10,000, what is the value of the land?
11. Use the Appendix tables at the end of the book for each of the following calculations:
    a. The cost of a new automobile is $10,000. If the interest rate is 5 percent, how much would you have to set aside now to provide this sum in five years?
    b. You have to pay $12,000 a year in school fees at the end of each of the next six years. If the interest rate is 8 percent, how much do you need to set aside today to cover these bills?
    c. You have invested $60,476 at 8 percent. After paying the above school fees, how much would remain at the end of the six years?
12. You have the opportunity to invest in the Belgravian Republic at 25 percent interest. The inflation rate is 21 percent. What is the real rate of interest?
13. The continuously compounded interest rate is 12 percent.
    a. You invest $1,000 at this rate. What is the investment worth after five years?
    b. What is the PV of $5 million to be received in eight years?
    c. What is the PV of a continuous stream of cash flows, amounting to $2,000 per year, starting immediately and continuing for 15 years?
14. You are quoted an interest rate of 6 percent on an investment of $10 million. What is the value of your investment after four years if the interest rate is compounded:
    a. Annually, **b.** monthly, or **c.** continuously?
15. Suppose the interest rate on five-year U.S. government bonds falls to 4.0 percent. Recalculate the value of the 7 percent bond maturing in 2006. (See Section 3.5.)
16. What is meant by a bond's yield to maturity and how is it calculated?

PRACTICE QUESTIONS

1. Use the *discount factors* shown in Appendix Table 1 at the end of the book to calculate the PV of $100 received in:
   a. Year 10 (at a discount rate of 1 percent).
   b. Year 10 (at a discount rate of 13 percent).
   c. Year 15 (at a discount rate of 25 percent).
   d. Each of years 1 through 3 (at a discount rate of 12 percent).
2. Use the *annuity factors* shown in Appendix Table 3 to calculate the PV of $100 in each of:
   a. Years 1 through 20 (at a discount rate of 23 percent).
   b. Years 1 through 5 (at a discount rate of 3 percent).
   c. Years 3 through 12 (at a discount rate of 9 percent).
3. **a.** If the one-year discount factor is .88, what is the one-year interest rate?
   b. If the two-year interest rate is 10.5 percent, what is the two-year discount factor?
   c. Given these one- and two-year discount factors, calculate the two-year annuity factor.
   d. If the PV of $10 a year for three years is $24.49, what is the three-year annuity factor?
   e. From your answers to (c) and (d), calculate the three-year discount factor.

EXCEL

4. A factory costs $800,000. You reckon that it will produce an inflow after operating costs of $170,000 a year for 10 years. If the opportunity cost of capital is 14 percent, what is the net present value of the factory? What will the factory be worth at the end of five years?

EXCEL

5. Harold Filbert is 30 years of age and his salary next year will be $20,000. Harold forecasts that his salary will increase at a steady rate of 5 percent per annum until his retirement at age 60.
   a. If the discount rate is 8 percent, what is the PV of these future salary payments?
   b. If Harold saves 5 percent of his salary each year and invests these savings at an interest rate of 8 percent, how much will he have saved by age 60?
   c. If Harold plans to spend these savings in even amounts over the subsequent 20 years, how much can he spend each year?

6. A factory costs $400,000. You reckon that it will produce an inflow after operating costs of $100,000 in year 1, $200,000 in year 2, and $300,000 in year 3. The opportunity cost of capital is 12 percent. Draw up a worksheet like that shown in Table 3.1 and use tables to calculate the NPV.

7. Halcyon Lines is considering the purchase of a new bulk carrier for $8 million. The forecasted revenues are $5 million a year and operating costs are $4 million. A major refit costing $2 million will be required after both the fifth and tenth years. After 15 years, the ship is expected to be sold for scrap at $1.5 million. If the discount rate is 8 percent, what is the ship's NPV?

EXCEL

8. As winner of a breakfast cereal competition, you can choose one of the following prizes:
   a. $100,000 now.
   b. $180,000 at the end of five years.
   c. $11,400 a year forever.
   d. $19,000 for each of 10 years.
   e. $6,500 next year and increasing thereafter by 5 percent a year forever.

   If the interest rate is 12 percent, which is the most valuable prize?

9. Refer back to the story of Ms. Kraft in Section 3.1.
   a. If the one-year interest rate were 25 percent, how many plays would Ms. Kraft require to become a millionaire? (*Hint:* You may find it easier to use a calculator and a little trial and error.)
   b. What does the story of Ms. Kraft imply about the relationship between the one-year discount factor, DF_1, and the two-year discount factor, DF_2?

10. Siegfried Basset is 65 years of age and has a life expectancy of 12 more years. He wishes to invest $20,000 in an annuity that will make a level payment at the end of each year until his death. If the interest rate is 8 percent, what income can Mr. Basset expect to receive each year?

11. James and Helen Turnip are saving to buy a boat at the end of five years. If the boat costs $20,000 and they can earn 10 percent a year on their savings, how much do they need to put aside at the end of years 1 through 5?

12. Kangaroo Autos is offering free credit on a new $10,000 car. You pay $1,000 down and then $300 a month for the next 30 months. Turtle Motors next door does not offer free credit but will give you $1,000 off the list price. If the rate of interest is 10 percent a year, which company is offering the better deal?

13. Recalculate the NPV of the office building venture in Section 3.1 at interest rates of 5, 10, and 15 percent. Plot the points on a graph with NPV on the vertical axis and the discount rates on the horizontal axis. At what discount rate (approximately) would the project have zero NPV? Check your answer.

14. **a.** How much will an investment of $100 be worth at the end of 10 years if invested at 15 percent a year simple interest?
b. How much will it be worth if invested at 15 percent a year compound interest?
c. How long will it take your investment to double its value at 15 percent compound interest?

15. You own an oil pipeline which will generate a $2 million cash return over the coming year. The pipeline's operating costs are negligible, and it is expected to last for a very long time. Unfortunately, the volume of oil shipped is declining, and cash flows are expected to decline by 4 percent per year. The discount rate is 10 percent.
a. What is the PV of the pipeline's cash flows if its cash flows are assumed to last forever?
b. What is the PV of the cash flows if the pipeline is scrapped after 20 years?

[*Hint for part (b):* Start with your answer to part (a), then subtract the present value of a declining perpetuity starting in year 21. Note that the forecasted cash flow for year 21 will be much less than the cash flow for year 1.]

16. If the interest rate is 7 percent, what is the value of the following three investments?
a. An investment that offers you $100 a year in perpetuity with the payment at the *end* of each year.
b. A similar investment with the payment at the *beginning* of each year.
c. A similar investment with the payment spread evenly over each year.

17. Refer back to Section 3.2. If the rate of interest is 8 percent rather than 10 percent, how much would our benefactor need to set aside to provide each of the following?
a. $100,000 at the end of each year in perpetuity.
b. A perpetuity that pays $100,000 at the end of the first year and that grows at 4 percent a year.
c. $100,000 at the end of each year for 20 years.
d. $100,000 a year spread evenly over 20 years.

18. For an investment of $1,000 today, the Tiburon Finance Company is offering to pay you $1,600 at the end of 8 years. What is the annually compounded rate of interest? What is the continuously compounded rate of interest?

19. How much will you have at the end of 20 years if you invest $100 today at 15 percent *annually* compounded? How much will you have if you invest at 15 percent *continuously* compounded?

20. You have just read an advertisement stating, "Pay us $100 a year for 10 years and we will pay you $100 a year thereafter in perpetuity." If this is a fair deal, what is the rate of interest?

21. Which would you prefer?
a. An investment paying interest of 12 percent compounded annually.
b. An investment paying interest of 11.7 percent compounded semiannually.
c. An investment paying 11.5 percent compounded continuously.

Work out the value of each of these investments after 1, 5, and 20 years.

22. Fill in the blanks in the following table:

Nominal Interest Rate (%)	Inflation Rate (%)	Real Interest Rate (%)
6	1	—
—	10	12
9	—	3

Visit us at www.mhhe.com/bm7e

23. Sometimes real rates of return are calculated by *subtracting* the rate of inflation from the nominal rate. This rule of thumb is a good approximation if the inflation rate is low. How big is the error from using this rule of thumb to calculate real rates of return in the following cases?

Nominal Rate (%)	Inflation Rate (%)
6	2
9	5
21	10
70	50

24. In 1880 five aboriginal trackers were each promised the equivalent of 100 Australian dollars for helping to capture the notorious outlaw Ned Kelley. In 1993 the granddaughters of two of the trackers claimed that this reward had not been paid. The prime minister of Victoria stated that, if this was true, the government would be happy to pay the $100. However, the granddaughters also claimed that they were entitled to compound interest. How much was each entitled to if the interest rate was 5 percent? What if it was 10 percent?

25. A leasing contract calls for an immediate payment of $100,000 and nine subsequent $100,000 semiannual payments at six-month intervals. What is the PV of these payments if the *annual* discount rate is 8 percent?

26. A famous quarterback just signed a $15 million contract providing $3 million a year for five years. A less famous receiver signed a $14 million five-year contract providing $4 million now and $2 million a year for five years. Who is better paid? The interest rate is 10 percent.

27. In August 1994 *The Wall Street Journal* reported that the winner of the Massachusetts State Lottery prize had the misfortune to be both bankrupt and in prison for fraud. The prize was $9,420,713, to be paid in 19 equal annual installments. (There were 20 installments, but the winner had already received the first payment.) The bankruptcy court judge ruled that the prize should be sold off to the highest bidder and the proceeds used to pay off the creditors. **a.** If the interest rate was 8 percent, how much would you have been prepared to bid for the prize? **b.** Enhance Reinsurance Company was reported to have offered $4.2 million. Use Appendix Table 3 to find (approximately) the return that the company was looking for.

28. You estimate that by the time you retire in 35 years, you will have accumulated savings of $2 million. If the interest rate is 8 percent and you live 15 years after retirement, what annual level of expenditure will those savings support?

Unfortunately, inflation will eat into the value of your retirement income. Assume a 4 percent inflation rate and work out a spending program for your retirement that will allow you to maintain a level *real* expenditure during retirement.

29. You are considering the purchase of an apartment complex that will generate a net cash flow of $400,000 per year. You normally demand a 10 percent rate of return on such investments. Future cash flows are expected to grow with inflation at 4 percent per year. How much would you be willing to pay for the complex if it:

a. Will produce cash flows forever?

b. Will have to be torn down in 20 years? Assume that the site will be worth $5 million at that time net of demolition costs. (The $5 million includes 20 years' inflation.)

Now calculate the real discount rate corresponding to the 10 percent nominal rate. Redo the calculations for parts (a) and (b) using real cash flows. (Your answers should not change.)

30. Vernal Pool, a self-employed herpetologist, wants to put aside a fixed fraction of her annual income as savings for retirement. Ms. Pool is now 40 years old and makes $40,000 a year. She expects her income to increase by 2 percentage points over inflation (e.g., 4 percent inflation means a 6 percent increase in income). She wants to accumulate $500,000 in real terms to retire at age 70. What fraction of her income does she need to set aside? Assume her retirement funds are conservatively invested at an expected real rate of return of 5 percent a year. Ignore taxes.

31. At the end of June 2001, the yield to maturity on U.S. government bonds maturing in 2006 was about 4.8 percent. Value a bond with a 6 percent coupon maturing in June 2006. The bond's face value is $10,000. Assume annual coupon payments and annual compounding. How does your answer change with semiannual coupons and a semiannual discount rate of 2.4 percent?

32. Refer again to Practice Question 31. How would the bond's value change if interest rates fell to 3.5 percent per year?

33. A two-year bond pays a coupon rate of 10 percent and a face value of $1,000. (In other words, the bond pays interest of $100 per year, and its principal of $1,000 is paid off in year 2.) If the bond sells for $960, what is its approximate yield to maturity? *Hint:* This requires some trial-and-error calculations.

CHALLENGE QUESTIONS

1. Here are two useful rules of thumb. The "Rule of 72" says that with discrete compounding the time it takes for an investment to double in value is roughly 72/interest rate (in percent). The "Rule of 69" says that with continuous compounding the time that it takes to double is *exactly* 69.3/interest rate (in percent).

   a. If the annually compounded interest rate is 12 percent, use the Rule of 72 to calculate roughly how long it takes before your money doubles. Now work it out exactly.

   b. Can you prove the Rule of 69?

2. Use a spreadsheet program to construct your own set of annuity tables.

3. An oil well now produces 100,000 barrels per year. The well will produce for 18 years more, but production will decline by 4 percent per year. Oil prices, however, will increase by 2 percent per year. The discount rate is 8 percent. What is the PV of the well's production if today's price is $14 per barrel?

4. Derive the formula for a growing (or declining) annuity.

5. Calculate the real cash flows on the 7 percent U.S. Treasury bond (see Section 3.5) assuming annual interest payments and an inflation rate of 2 percent. Now show that by discounting these real cash flows at the real interest rate you get the same PV that you get when you discount the nominal cash flows at the nominal interest rate.

6. Use a spreadsheet program to construct a set of bond tables that shows the present value of a bond given the coupon rate, maturity, and yield to maturity. Assume that coupon payments are semiannual and yields are compounded semiannually.

MINI-CASE

The Jones Family, Incorporated

The Scene: Early evening in an ordinary family room in Manhattan. Modern furniture, with old copies of *The Wall Street Journal* and the *Financial Times* scattered around. Autographed photos of Alan Greenspan and George Soros are prominently displayed. A picture window reveals a distant view of lights on the Hudson River. John Jones sits at a computer terminal, glumly sipping a glass of chardonnay and trading Japanese yen over the Internet. His wife Marsha enters.

Marsha: Hi, honey. Glad to be home. Lousy day on the trading floor, though. Dullsville. No volume. But I did manage to hedge next year's production from our copper mine. I couldn't get a good quote on the right package of futures contracts, so I arranged a commodity swap.

John doesn't reply.

Marsha: John, what's wrong? Have you been buying yen again? That's been a losing trade for weeks.

John: Well, yes. I shouldn't have gone to Goldman Sachs's foreign exchange brunch. But I've got to get out of the house somehow. I'm cooped up here all day calculating covariances and efficient risk-return tradeoffs while you're out trading commodity futures. You get all the glamour and excitement.

Marsha: Don't worry dear, it will be over soon. We only recalculate our most efficient common stock portfolio once a quarter. Then you can go back to leveraged leases.

John: You trade, and I do all the worrying. Now there's a rumor that our leasing company is going to get a hostile takeover bid. I knew the debt ratio was too low, and you forgot to put on the poison pill. And now you've made a negative-NPV investment!

Marsha: What investment?

John: Two more oil wells in that old field in Ohio. You spent $500,000! The wells only produce 20 barrels of crude oil per day.

Marsha: That's 20 barrels day in, day out. There are 365 days in a year, dear.

John and Marsha's teenage son Johnny bursts into the room.

Johnny: Hi, Dad! Hi, Mom! Guess what? I've made the junior varsity derivatives team! That means I can go on the field trip to the Chicago Board Options Exchange. *(Pauses.)* What's wrong?

John: Your mother has made another negative-NPV investment. More oil wells.

Johnny: That's OK, Dad. Mom told me about it. I was going to do an NPV calculation yesterday, but my corporate finance teacher asked me to calculate default probabilities for a sample of junk bonds for Friday's class.

(Grabs a financial calculator from his backpack.) Let's see: 20 barrels per day times $15 per barrel times 365 days per year . . . that's $109,500 per year.

John: That's $109,500 *this* year. Production's been declining at 5 percent every year.

Marsha: On the other hand, our energy consultants project increasing oil prices. If they increase with inflation, price per barrel should climb by roughly 2.5 percent per year. The wells cost next to nothing to operate, and they should keep pumping for 10 more years at least.

Johnny: I'll calculate NPV after I finish with the default probabilities. Is a 9 percent nominal cost of capital OK?

Marsha: Sure, Johnny.

John: *(Takes a deep breath and stands up.)* Anyway, how about a nice family dinner? I've reserved our usual table at the Four Seasons.

Everyone exits.

Announcer: Were the oil wells really negative-NPV? Will John and Marsha have to fight a hostile takeover? Will Johnny's derivatives team use Black-Scholes or the binomial method? Find out in the next episode of The Jones Family, Incorporated.

You may not aspire to the Jones family's way of life, but you will learn about all their activities, from futures contracts to binomial option pricing, later in this book. Meanwhile, you may wish to replicate Johnny's NPV analysis.

Questions

1. Forecast future cash flows, taking account of the decline in production and the (partially) offsetting forecasted increase in oil prices. How long does production have to continue for the oil wells to be a positive-NPV investment? You can ignore taxes and other possible complications.